



## Fast Engset computation

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### ABSTRACT

The blocking probability of a finite-source bufferless queue is a fixed point of the Engset formula, for which we prove existence and uniqueness. Numerically, the literature suggests a fixed point iteration. We show that such an iteration can fail to converge and is dominated by a simple Newton's method, for which we prove a global convergence result. The analysis yields a new Turán-type inequality involving hypergeometric functions, which is of independent interest.

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### 1. Introduction

The Engset formula is used to determine the blocking probability in a bufferless queueing system with a finite population of sources. Applications to *bufferless optical networks* [6,20,12,14,13] have sparked a renewed interest in the Engset model and its generalizations [5]. Sztrik provides a literature review of applications [18], including *multiprocessor performance modeling* and the *machine interference problem*, in which machines request service from one or more repairmen. The analysis herein was inspired by a recent application in *sizing vehicle pools* for car-shares [4].

The queue under consideration is the  $M/M/m/m/N$  queue [10]. This is a bufferless queue with  $N$  sources that can request service, provided by one of  $m$  identical servers. When all  $m$  servers are in use, incoming arrivals are *blocked* and leave the system without queueing. The *Engset formula* is used to determine the probability  $P$  that any random arrival is blocked. The Engset formula is [11, Equation (62)]

$$P = \lim_{P' \rightarrow P} \underbrace{\frac{\binom{N-1}{m} (M(P'))^m}{\sum_{X=0}^m \binom{N-1}{X} (M(P'))^X}}_{f(P)} \quad \text{where } M(P) = \frac{\alpha}{1 - \alpha(1 - P)} \quad (\text{Engset formula})$$

The number of sources  $N$ , the number of servers  $m$ , and the offered traffic *per-source*  $\alpha$  are given as input. It is worthwhile to note that subject to some technical assumptions, the Engset formula remains valid under general distributions (i.e.  $G/G/m/m/N$ ) [19, Section 5.4].

It is not obvious if any value of  $P$  satisfies the *Engset formula*, or if multiple values of  $P$  might satisfy it. To the authors' best knowledge, this work is the first to establish the existence and uniqueness of a solution (Section 2).

**Remark.** The limit appearing in the *Engset formula* is a technical detail to avoid (for ease of analysis) the removable discontinuity at  $P = 1 - 1/\alpha$ . We mention that  $f$  may admit nonremovable discontinuities at some negative values of  $P$  (at which the limit does not exist), though this does not affect the analysis.

**Remark.** Let  $\lambda$  be the *idle source initiation rate*, the rate at which a free source (i.e. one not being serviced) initiates requests, and  $1/\mu$  be the *mean service time*. If  $P$  is the blocking probability,  $M(P) = \lambda/\mu$ . This substitution removes  $P$  from the right-hand side of the *Engset formula* [11, Equation (70)]. However,  $\lambda$  is often unknown in practice, and hence this method is only applicable in special cases, or subject to error produced from approximating  $\lambda$ .

### 2. Properties of the Engset formula

If the number of servers  $m$  is zero, any request entering the queue is blocked ( $P = 1$ ). If there are at least as many servers as there are sources ( $m \geq N$ ), any request entering the queue can

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immediately be serviced ( $P = 0$ ). Finally, the case of zero traffic ( $\alpha = 0$ ) corresponds to a queue that receives no requests. We assume the following for the remainder of this work:

**Assumption.**  $m$  and  $N$  are integers with  $0 < m < N$ .  $\alpha$  is a positive real number.

The following lemmas characterize  $f$  defined in the [Engset formula](#) and are used to establish several results throughout this work:

**Lemma 1.**  $f$  is strictly decreasing on  $[0, \infty)$ .

**Lemma 2.**  $f$  is convex on  $[1 - 1/\alpha, \infty) \supset [1, \infty)$ .

Owing partly to [Lemma 1](#), our first significant result is as follows:

**Theorem 3.** There exists a unique probability  $P^*$  satisfying the [Engset formula](#).

Proofs of these results are given in the [Appendix](#). The proof of [Theorem 3](#) establishes that  $f(0) - 0$  and  $f(1) - 1$  have opposite signs. Therefore,  $P^*$  can be computed via the bisection method on the interval  $[0, 1]$  applied to the map

$$P \mapsto f(P) - P. \tag{1}$$

### 3. Computation

#### 3.1. Fixed point iteration

The literature suggests the use of a fixed point iteration [[9](#), page 489]. This involves picking an initial guess  $P_0$  for the blocking probability and considering the iterates of  $f$  evaluated at  $P_0$ . Specifically,

$$P_0 \in [0, 1]$$

$$P_n = f(P_{n-1}) \quad \text{for } n > 0. \tag{fixed point iteration}$$

We characterize convergence in the following result:

**Theorem 4.** If  $\alpha \leq 1$  and  $|f'(0)| < 1$ , the [fixed point iteration](#) converges to  $P^*$ .

While the first inequality appearing above is a restriction on the per-source traffic, the second inequality is hard to verify, as it involves the derivative of  $f$ . This inspires the following:

**Corollary 5.** If  $\alpha \leq 1$  and  $N \geq 2m$ , the [fixed point iteration](#) converges to  $P^*$ .

The condition  $N \geq 2m$  requires there to be twice as many sources as there are servers, satisfied in most (but not all) reasonable queueing systems.

Proofs of these results are given in the [Appendix](#).

#### 3.2. Newton's method

*Newton's method* uses first-derivative information in an attempt to speed up convergence. In particular,

$$P_0 \in [0, 1]$$

$$P_n = P_{n-1} - \frac{f(P_{n-1}) - P_{n-1}}{f'(P_{n-1}) - 1} \quad \text{for } n > 0. \tag{Newton's method}$$

Often, convergence results for applications of Newton's method are *local* in nature: they depend upon the choice of initial guess  $P_0$ . By using the convexity established in [Lemma 2](#), we are able to derive a *global* result for Newton's method:

**Theorem 6.** If  $\alpha \leq 1$ , *Newton's method* converges to  $P^*$ .

**Table 1**  
Comparison under  $N = 20$  and  $\alpha = \frac{1}{4}$ .

Servers $m$	Probability $P^*$	Number of iterations	
		Fixed point	Newton
1	8.322e-01	6	3
2	6.725e-01	7	3
3	5.235e-01	7	3
4	3.879e-01	8	3
5	2.693e-01	9	3
6	1.714e-01	8	4
7	9.718e-02	8	4
8	4.753e-02	7	4
9	1.947e-02	6	4
10	6.554e-03	5	3
11	1.798e-03	4	3
12	4.005e-04	4	3
13	7.194e-05	3	3
14	1.028e-05	3	3
15	1.142e-06	3	3
16	9.518e-08	3	2
17	5.599e-09	2	2
18	2.074e-10	2	2
19	3.638e-12	2	2

**Table 2**  
Comparison under  $N = 20$  and  $\alpha = \frac{1}{2}$ .

Servers $m$	Probability $P^*$	Number of iterations	
		Fixed point	Newton
1	9.087e-01	7	3
2	8.187e-01	8	3
3	7.303e-01	9	3
4	6.436e-01	10	3
5	5.591e-01	11	3
6	4.773e-01	11	3
7	3.985e-01	14	3
8	3.235e-01	15	4
9	2.531e-01	16	4
10	1.885e-01	16	4
11	1.310e-01	14	4
12	8.259e-02	12	4
13	4.527e-02	10	4
14	2.041e-02	8	4
15	7.124e-03	6	4
16	1.827e-03	5	4
17	3.254e-04	4	3
18	3.623e-05	3	3
19	1.907e-06	3	3

A proof of this result is given in the [Appendix](#). Superficially, [Theorem 6](#) seems preferable to [Corollary 5](#) as it does not place restrictions on  $N$  or  $m$ . In practice, we will see that [Newton's method](#) outperforms the [fixed point iteration](#), and that it performs well even when  $\alpha > 1$  ([Section 4](#)).

### 4. Comparison of methods

[Tables 1–4](#) compare the performance the [fixed point iteration](#) and [Newton's method](#) for a queueing system with  $N = 20$  sources (though we mention that the observed trends seem to hold independent of our choice of  $N$ ). The initial guess used is  $P_0 = \frac{1}{2}$ . The stopping criterion used is  $|P_{n+1} - P_n| \leq tol = 2^{-24}$ .

Bisection halves the search interval at each step, so that the maximum possible error at the  $n$ th iteration is  $2^{-n}$ . It follows that to achieve a desired error tolerance  $tol$ , bisection requires  $\lceil -\lg(tol) \rceil = \lceil -\lg(2^{-24}) \rceil = 24$  iterations independent of the input parameters (for this reason, it is omitted from the tables). The [fixed point iteration](#) fails to converge or performs poorly (sometimes taking hundreds of iterations) precisely when the sufficient conditions of [Corollary 5](#) are violated. [Newton's method](#) outperforms both algorithms by a wide margin, often converging in just a few iterations.

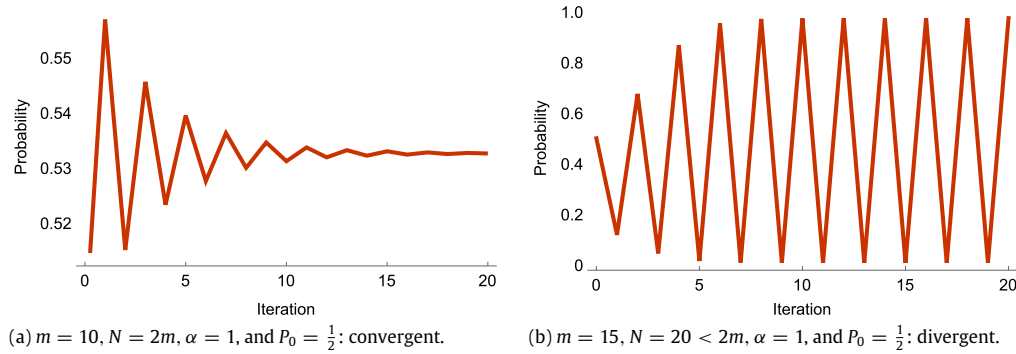


Fig. 1. Oscillatory nature of the fixed point iteration.

Table 3  
Comparison under  $N = 20$  and  $\alpha = 1$ . FAIL indicates divergence.

Servers $m$	Probability $P^*$	Number of iterations	
		Fixed point	Newton
1	9.523e-01	7	3
2	9.047e-01	8	3
3	8.574e-01	10	3
4	8.102e-01	12	3
5	7.633e-01	14	4
6	7.166e-01	17	4
7	6.702e-01	20	4
8	6.241e-01	25	4
9	5.782e-01	33	4
10	5.327e-01	45	3
11	4.874e-01	79	3
12	4.424e-01	556	4
13	3.976e-01	FAIL	4
14	3.530e-01	FAIL	4
15	3.084e-01	FAIL	5
16	2.636e-01	FAIL	5
17	2.181e-01	FAIL	6
18	1.708e-01	FAIL	7
19	1.187e-01	FAIL	7

Table 4  
Comparison under  $N = 20$  and  $\alpha = 2$ . FAIL indicates divergence.

Servers $m$	Probability $P^*$	Number of iterations	
		Fixed point	Newton
1	9.756e-01	7	3
2	9.512e-01	9	3
3	9.268e-01	10	3
4	9.025e-01	13	4
5	8.781e-01	15	4
6	8.538e-01	19	4
7	8.295e-01	24	4
8	8.053e-01	33	4
9	7.810e-01	54	4
10	7.568e-01	136	4
11	7.325e-01	FAIL	4
12	7.083e-01	FAIL	4
13	6.840e-01	FAIL	4
14	6.597e-01	FAIL	4
15	6.353e-01	FAIL	4
16	6.107e-01	FAIL	4
17	5.859e-01	FAIL	5
18	5.604e-01	FAIL	5
19	5.336e-01	FAIL	5

Insight into the poor performance of the fixed point iteration is given by Corollary 12 of Appendix, which exploits the oscillatory nature of the fixed point iteration (see Fig. 1) to derive successively tighter upper bounds on the number of iterations required for convergence up to a desired error tolerance.

**Remark.** Naïve implementations computing  $f$  (and  $f'$ ) directly may take more iterations than necessary due to floating point error. Lemma 9 of Appendix shows that  $f$  is a reciprocal of a hypergeometric function so that standard computational techniques [16] can be used. A quasi-Newton implementation has been made available by the authors: <https://github.com/parsiad/fast-engset/releases>.

### 5. A Turán-type inequality

Turán-type inequalities are named after Paul Turán, who proved the result  $(L_n(x))^2 > L_{n-1}(x)L_{n+1}(x)$  on  $-1 < x < 1$  for the Legendre Polynomials  $\{L_n\}$ . Such inequalities appear frequently for hypergeometric functions and are often a direct consequence of their log-concavity/convexity. There exists a maturing body of work characterizing the log-concavity/convexity and associated Turán-type inequalities of generalized hypergeometric functions (see, e.g., [2,3,8,7]).

The analysis used to prove Lemma 2 gives rise to a new Turán-type inequality. Letting  ${}_2F_1$  denote the ordinary hypergeometric function [1], we have the following result, whose proof is given in the Appendix:

**Theorem 7 (A Turán-type Inequality).** Let  $b$  be a positive integer,  $c$  a positive real number, and

$$h_n(x) = {}_2F_1(1 + n, -b + n; c + n; -x).$$

Then, the map  $x \mapsto h_1(x)/(h_0(x))^2$  is strictly decreasing on  $[0, \infty)$  and

$$b(c + 1) \cdot (h_1(x))^2 \geq (b - 1)c \cdot h_0(x)h_2(x) \quad \text{for } x \geq 0. \quad (2)$$

### 6. Future work

Numerical evidence suggests that Lemma 2 can be relaxed:

**Conjecture 8.**  $f$  is convex on  $[0, \infty)$ .

This result would remove the requirement  $\alpha \leq 1$  from all claims in this work. In particular, this would yield unconditional convergence for Newton's method.

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**Appendix. Proofs of results**

Let  $(\cdot)_X$  denote the Pochhammer symbol:

$$(c)_X = \begin{cases} c(c+1)\cdots(c+X-1), & \text{if } X \text{ is a positive integer;} \\ 1, & \text{if } X = 0. \end{cases} \quad \text{(Pochhammer symbol)}$$

The ordinary hypergeometric function [1] satisfies

$${}_2F_1(a, b; c; z) = \sum_{X=0}^{\infty} \frac{(a)_X (b)_X z^X}{(c)_X X!}$$

if  $b \in \{-1, -2, \dots\}$  or  $|z| < 1$ . (hypergeometric function)

The Pochhammer symbol can also be used to represent the falling factorial  $c^{(X)}$ :

$$\begin{aligned} c^{(X)} &= c(c-1)\cdots(c-X+1) \\ &= (-c)(-1)(-c+1)(-1)\cdots(-c+X-1)(-1) \\ &= (-c)_X (-1)^X. \end{aligned}$$

**Lemma 9.**  $f(P)$  defined in the Engset formula satisfies

$$1/f(P) = {}_2F_1(1, -m; N - m; 1 - P - 1/\alpha).$$

**Proof.** If  $P = 1 - 1/\alpha$ , the claim is trivial. Otherwise, the reciprocal of  $M(P)$  in the Engset formula is

$$1/M(P) = -(1 - P - 1/\alpha). \quad \text{(A.1)}$$

We can write the binomial coefficients in the Engset formula in terms of Pochhammer symbols as follows:

$$\binom{N-1}{X} / \binom{N-1}{m} = \frac{m! (N-1-m)!}{X! (N-1-X)!} = \frac{m^{(m-X)}}{(N-m)_{m-X}}. \quad \text{(A.2)}$$

Substituting (A.1) and (A.2) into the reciprocal of  $f(P)$  yields

$$\begin{aligned} \frac{1}{f(P)} &= \sum_{X=0}^m \frac{m^{(m-X)}}{(N-m)_{m-X}} (1/M(P))^{m-X} \\ &= \sum_{X=0}^m \frac{m^{(X)}}{(N-m)_X} (1/M(P))^X \\ &= \sum_{X=0}^{\infty} \frac{(-m)_X}{(N-m)_X} (1 - P - 1/\alpha)^X. \end{aligned}$$

The upper bound of summation is relaxed to  $\infty$  in the last equality since  $(-m)_X = 0$  if  $X > m$ . The desired result then follows from multiplying each summand in the series by  $(1)_X/X! = 1$ .  $\square$

The following identity should be understood subject to the convention  $0 \cdot \infty = \infty \cdot 0$  ( $\infty$  denotes complex infinity):

**Lemma 10 (Hypergeometric Binomial Theorem).** Suppose  $b$  is a negative integer and  $c$  is not an integer satisfying  $b \leq c \leq 0$ . Then,

$${}_2F_1(a, b; c; z+w) = \sum_{Y=0}^{\infty} \frac{(a)_Y (b)_Y z^Y}{(c)_Y Y!} {}_2F_1(a+Y, b+Y; c+Y; w).$$

**Proof.** An application of the binomial theorem yields

$$\begin{aligned} {}_2F_1(a, b, c; z+w) &= \sum_{X=0}^{\infty} \frac{(a)_X (b)_X (z+w)^X}{(c)_X X!} \\ &= \sum_{X=0}^{\infty} \frac{(a)_X (b)_X}{(c)_X} \frac{1}{X!} \sum_{Y=0}^X \binom{X}{Y} z^Y w^{X-Y} \end{aligned}$$

$$\begin{aligned} &= \sum_{Y=0}^{\infty} \frac{z^Y}{Y!} \sum_{X=Y}^{\infty} \frac{(a)_X (b)_X}{(c)_X} \frac{w^{X-Y}}{(X-Y)!} \\ &= \sum_{Y=0}^{\infty} \frac{(a)_Y (b)_Y z^Y}{(c)_Y Y!} \sum_{X=Y}^{\infty} \frac{(a+Y)_{X-Y} (b+Y)_{X-Y}}{(c+Y)_{X-Y}} \frac{w^{X-Y}}{(X-Y)!}. \end{aligned}$$

The desired result follows by shifting the index of summation to  $X = 0$ .  $\square$

**Lemma 10** can also be extended to the case where  $b$  is not a negative integer, but care must be taken to ensure that the various power series are convergent.

**Proof of Lemma 1.** To establish this, we show that  $P \mapsto 1/f(P)$  is a polynomial with positive coefficients. That is,

$$\frac{1}{f(P)} = \sum_{Y=0}^m c_Y P^Y \quad \text{where } c_Y > 0. \quad \text{(A.3)}$$

An application of Lemma 10 to the form in Lemma 9 reveals that

$$c_Y = \frac{m^{(Y)}}{(N-m)_Y} d_Y. \quad \text{(A.4)}$$

where  $d_Y = {}_2F_1(1+Y, -(m-Y); N-m+Y; 1-1/\alpha)$ . To arrive at (A.3), it suffices to show  $d_Y > 0$ . Another application of Lemma 10 along with the identity

$${}_2F_1(a, -b; c; 1) = \frac{(c-a)_b}{(c)_b} \quad \text{if } b \text{ is a nonnegative integer}$$

yields

$$d_Y = \sum_{Z=0}^{m-Y} \frac{(1/\alpha)^Z (1+Y)_Z (m-Y)^{(Z)}}{Z! (N-m+Y)_Z (N-m+Y+Z)_{m-Y-Z}}, \quad \text{(A.5)}$$

which is trivially positive.  $\square$

The following is found in [8, Lemma 1]:

**Lemma 11.** Let

$$A(Q) = \sum_{X=0}^N a_X Q^X \quad \text{and} \quad B(Q) = \sum_{X=0}^N b_X Q^X$$

be distinct polynomials with nonnegative coefficients satisfying  $a_X b_{X-1} \leq a_{X-1} b_X$  for  $0 < X \leq N$  and  $b_X > 0$  for  $0 \leq X \leq N$ . Then, the map  $Q \mapsto A(Q)/B(Q)$  is strictly decreasing on  $[0, \infty)$ .

**Proof of Lemma 2.** The derivative of the hypergeometric function is

$$\frac{\partial}{\partial z} {}_2F_1(a, b; c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1; c+1; z). \quad \text{(A.6)}$$

This fact combined with the representation in Lemma 9 yields

$$f'(P) = -\frac{m}{N-m} \frac{A(P+1/\alpha-1)}{B(P+1/\alpha-1)} \quad \text{(A.7)}$$

where

$$A(Q) = {}_2F_1(2, -(m-1); N-m+1; -Q)$$

$$\text{and } B(Q) = ({}_2F_1(1, -m; N-m; -Q))^2.$$

To arrive at the desired result, we seek to show that the map

$$Q \mapsto A(Q)/B(Q) \quad \text{(A.8)}$$

is strictly decreasing on  $[0, \infty)$ .

For notational succinctness, let  $S = N - m$ . We can write Eq. (A.8) as a quotient of polynomials by noting that

$$A(Q) = \sum_{x=0}^{m-1} \frac{(X+1)(m-1)^{(x)}}{(S+1)_x} Q^x$$

and (expanding using the Cauchy product)

$$B(Q) = \left( \sum_{x=0}^m \frac{m^{(x)}}{(S)_x} Q^x \right)^2 = \sum_{x=0}^{2m} Q^x \sum_{y=0}^x \frac{m^{(y)}}{(S)_y} \frac{m^{(x-y)}}{(S)_{x-y}}$$

We seek to apply Lemma 11 on the polynomials  $A$  and  $B$ , whose coefficients we denote  $a_x$  and  $b_x$ , respectively. Note that  $A$  and  $B$  are distinct since  $0 = a_x < b_x$  for  $m \leq X \leq 2m$ . One can easily check that  $a_1 = a_1 b_0 \leq a_0 b_1 = b_1$ . We thus need only verify  $a_x b_{x-1} \leq a_{x-1} b_x$  for  $X > 1$ .

Fix  $X > 1$ . It is easy to check that

$$a_x = a_{x-1} \left( \frac{1}{X} + 1 \right) \frac{m-X}{S+X}$$

Using Gauss summation, we can rewrite  $b_x$  as

$$b_x = \mathbf{1}_{\{x \text{ is even}\}} \left( \frac{m^{(x/2)}}{(S)_{x/2}} \right)^2 + 2 \sum_{y=0}^{\lfloor (x-1)/2 \rfloor} \frac{m^{(y)}}{(S)_y} \frac{m^{(x-y)}}{(S)_{x-y}}$$

Suppose  $X$  is even. Then,

$$\begin{aligned} a_x b_{x-1} &= 2a_{x-1} \left( \frac{1}{X} + 1 \right) \frac{m-X}{S+X} \left( \sum_{y=0}^{X/2-1} \frac{m^{(y)}}{(S)_y} \frac{m^{(x-y-1)}}{(S)_{x-y-1}} \right) \\ &\leq 2a_{x-1} \left( \frac{1}{X} \frac{m-X}{S+X} \sum_{y=0}^{X/2-1} \left( \frac{m^{(X/2-1)}}{(S)_{X/2-1}} \frac{m^{(X/2)}}{(S)_{X/2}} \right) \right. \\ &\quad \left. + \sum_{y=0}^{X/2-1} \frac{m^{(y)}}{(S)_y} \frac{m^{(x-y)}}{(S)_{x-y}} \right) \\ &\leq a_{x-1} \left( \left( \frac{m^{(X/2)}}{(S)_{X/2}} \right)^2 + 2 \sum_{y=0}^{X/2-1} \frac{m^{(y)}}{(S)_y} \frac{m^{(x-y)}}{(S)_{x-y}} \right) \leq a_{x-1} b_x. \end{aligned}$$

A similar approach can be taken if  $X$  is odd. □

**Proof of Theorem 3.** We first show that  $f(1) < 1$ , or equivalently,  $1/f(1) > 1$ . By the positivity of (A.5), the map  $\alpha \mapsto 1/f(P; \alpha)$  is strictly decreasing. Passing to the limit and dropping higher order terms involving  $1/\alpha^z$  with  $Z > 0$  yields

$$\begin{aligned} \frac{1}{f(P; \alpha)} &> \lim_{\alpha' \rightarrow \infty} \frac{1}{f(P; \alpha')} \\ &= \sum_{y=0}^m P^y \frac{(m)^{(y)}}{(N-m)_y} \frac{(N-m-1)_{m-y}}{(N-m+Y)_{m-y}}. \end{aligned}$$

One can verify that if  $P = 1$ , the above sum is exactly one, yielding  $1/f(1) > 1$  (for all  $0 < \alpha < \infty$ ), as desired.

By Lemma 1, the map (1) is strictly decreasing on  $[0, \infty)$ . Furthermore, since  $f(1) < 1$ ,  $f(1) - 1$  and  $f(0) - 0 > 0$  have opposite signs. Because (1) is also continuous, the desired result follows by the intermediate value theorem. □

**Proof of Theorem 4.** Let  $I = [0, 1]$ . (A.3) establishes that  $f$  is positive on  $I$ . Since  $\alpha \leq 1$ ,  $d_0$  appearing in (A.4) satisfies  $d_0 \geq 1$ . It follows that  $f(0) = 1/d_0 \leq 1$  (see (A.3)). Letting  $I = [0, 1]$ , these facts and Lemma 1 yield  $f(I) \subset I$ . Since  $f$  is continuously differentiable on  $I$ , it suffices to show that there exists a nonnegative constant  $L < 1$  such that  $|f'| \leq L$  on  $I$  (implying that  $f$  is a contraction on  $I$ ).

Since  $P + 1/\alpha - 1 \geq 0$  whenever  $\alpha \leq 1$ , (A.7) reveals that  $-f' = |f'|$  on  $I$ . Owing to Lemma 2,  $f$  is convex on  $I$  so that  $-f'$  is nonincreasing on  $I$ . Therefore,  $|f'(0)| \geq |f'|$  on  $I$ , and the desired result follows by taking  $L = |f'(0)|$ . □

**Proof of Corollary 5.** We begin by considering the case of  $\alpha < 1$ ;  $\alpha = 1$  is handled separately. Recall that the proof of Lemma 2 shows that map (A.8) is strictly decreasing on  $[0, \infty)$ . Since  $1/\alpha - 1 > 0$  and  ${}_2F_1(a, b; c; 0) = 1$ ,

$$|f'(0)| = \frac{m}{N-m} \frac{A(1/\alpha - 1)}{B(1/\alpha - 1)} < \frac{m}{N-m} \frac{A(0)}{B(0)} = \frac{m}{N-m}, \quad (\text{A.9})$$

and the desired result follows ( $N \geq 2m$  is equivalent to  $m/(N-m) \leq 1$ ).

Suppose now  $\alpha = 1$ . We modify our approach, as the strict inequality in (A.9) no longer holds. By (A.3) and (A.4),

$$f(0) = {}_1/2F_1(1, -m, N-m; 0) = 1.$$

This along with the fact that  $f$  is strictly decreasing (Lemma 1) and  $0 < f(1) < 1$  implies that the iterates of  $f$  evaluated at some probability  $P_0$  (i.e.  $f^k(P_0)$  for  $k > 0$ ) reside in  $[f(1), 1]$ . We can thus relax the sufficient condition for convergence in Theorem 4 to  $|f'(f(1))| < 1$  in lieu of  $|f'(0)| < 1$ . Then

$$\frac{m}{N-m} \frac{A(f(1))}{B(f(1))} < \frac{m}{N-m} \frac{A(0)}{B(0)} = \frac{m}{N-m},$$

and the desired result follows. □

Let  $f^k$  denote the  $k$ th iterate of  $f$ . The proof above reveals that we can replace the condition  $|f'(0)| < 1$  with  $|f'(f^{2k}(0))| < 1$  for some nonnegative integer  $k$ . Owing to this, we derive a relaxation of Theorem 4 along with a family of bounds (parameterized by  $k$ ) on the number of iterations required for convergence up to a desired error tolerance  $\epsilon$ :

**Corollary 12.** Let  $k$  be a nonnegative integer and  $P^*$  denote the solution of the Engset formula. Suppose  $\alpha \leq 1$  and

$$q = |f'(f^{2k}(0))| < 1.$$

Given  $0 < \epsilon \leq 1$  and  $\{P_n\}$  as defined by the fixed point iteration,  $|P_{2k+\ell} - P^*| \leq \epsilon$  whenever

$$\ell \geq \lceil \log_q(\epsilon - \epsilon q) \rceil.$$

**Proof.** (A.3) establishes  $f(0) > 0$  and  $f^2(0) > 0$ . Using the fact that  $f$  is strictly decreasing (Lemma 1), it follows by induction that  $[0, 1] \supset [f^0(0), f^1(0)] \supseteq [f^2(0), f^3(0)] \supseteq \dots$

and  $P_{2k+\ell}$  is in the interval  $[f^{2k}(0), f^{2k+1}(0)]$  for all  $\ell \geq 0$ . The contraction mapping principle [15] characterizes the speed of convergence:

$$|P_{2k+\ell} - P^*| \leq \frac{q^\ell}{1-q} |P_{2k+1} - P_{2k}| \leq \frac{q^\ell}{1-q} \quad \text{for } \ell > 0.$$

The desired result follows. □

The proof of Theorem 6 requires the following result (a simple modification of [17, chapter 22, exercise 14b]):

**Lemma 13.** Let  $I$  be an interval and  $g: I \rightarrow \mathbb{R}$  be a convex and differentiable function satisfying  $g' < 0$  and  $g(x^*) = 0$  for some  $x^*$  in  $I$ . Then, given  $x_0 \in I$  with  $g(x_0) \geq 0$ , the sequence  $\{x_n\}$  defined by

$$x_n = x_{n-1} - g(x_{n-1})/g'(x_{n-1}) \quad \text{for } n > 0$$

converges from below (i.e.  $x_0 \leq x_1 \leq \dots$ ) to  $x^*$ .

**Proof.** Since  $g(x_0) \geq 0$  and  $g'(x_0) < 0$ , it follows that  $x_0 \leq x_1$ . Since  $(x_1, 0)$  is on a tangent line of  $g$  and a convex function lies above its tangent lines,  $g(x_1) \geq 0$ . Hence,  $x_1 \leq x^*$ . Repeating this argument establishes  $x_0 \leq x_1 \leq \dots \leq x^*$ .

It follows that  $x_n \rightarrow x$  for some  $x$  in  $I$ . Taking limits on both sides of  $g'(x_{n-1})(x_{n-1} - x_n) = g(x_{n-1})$  and using the facts that  $g$  is continuous and  $g'$  is monotone due to the assumption of convexity, we arrive at  $g(x) = 0$ . Since a strictly decreasing function cannot have two distinct roots,  $x = x^*$ . □

**Proof of Theorem 6.** First, consider the case  $f(P_0) - P_0 \geq 0$ . Lemma 1 implies that  $f' \leq 0$  and hence  $f' - 1 < 0$  on  $I = [0, 1]$ . Lemma 2 establishes that  $f$  is convex on  $I$ . Theorem 3 guarantees the existence of  $P^*$  in  $I$  such that  $f(P^*) - P^* = 0$ . Letting  $g: I \rightarrow \mathbb{R}$  be defined by  $g(P) = f(P) - P$ , we can directly apply Lemma 13.

Now, consider the case of  $f(P_0) - P_0 < 0$ . Note that

$$P_1 = P_0 - \frac{f(P_0) - P_0}{f'(P_0) - 1} = \frac{f(P_0) + P_0 |f'(P_0)|}{1 + |f'(P_0)|} > 0.$$

Since the point  $(P_1, 0)$  is on a tangent line of  $P \mapsto f(P) - P$  and a convex function lies above its tangent lines,  $f(P_1) - P_1 \geq 0$ . We can now repeat the argument in the first paragraph with the initial guess  $P_1$  in lieu of  $P_0$ .  $\square$

**Proof of Theorem 7.** That  $x \mapsto h_1(x)/(h_0(x))^2$  is strictly decreasing follows directly from the proof of Lemma 2. The derivative of this map is  $C(x)/(h_0(x))^3$  where

$$\begin{aligned} C(x) &= -2h_1(x)h_0'(x) + h_0(x)h_1'(x) \\ &= -\frac{2b}{c}(h_1(x))^2 + \frac{2(b-1)}{c+1}h_0(x)h_2(x) \end{aligned}$$

(the last equality is a consequence of (A.6)). Since  $h_0$  is positive on  $H = [0, \infty)$ , it follows that  $C$  is nonpositive on  $H$ , yielding (2).  $\square$

## References

- [1] M. Abramowitz, I.A. Stegun, *Handbook of Mathematical Functions*. Vol. 1, Dover Publications, 1965.
- [2] Á Baricz, Functional inequalities involving Bessel and modified Bessel functions of the first kind, *Expo. Math.* 26 (3) (2008) 279–293.
- [3] Á Baricz, Turán type inequalities for hypergeometric functions, *Proc. Amer. Math. Soc.* 136 (9) (2008) 3223–3229.
- [4] T. Carpenter, S. Keshav, J. Wong, Sizing finite-population vehicle pools, *IEEE Trans. Intell. Transp. Syst.* 15 (3) (2014) 1134–1144.
- [5] J.W. Cohen, The generalized Engset formulae, *Philips Telecommun. Rev.* 18 (4) (1957) 158–170.
- [6] A. Detti, V. Eramo, M. Listanti, Performance evaluation of a new technique for IP support in a WDM optical network: optical composite burst switching (OCBS), *J. Lightwave Technol.* 20 (2) (2002) 154–165.
- [7] S.I. Kalmykov, D.B. Karp, Log-convexity and log-concavity for series in gamma ratios and applications, *J. Math. Anal. Appl.* 406 (2) (2013) 400–418.
- [8] D.B. Karp, S.M. Sitnik, Log-convexity and log-concavity of hypergeometric-like functions, *J. Math. Anal. Appl.* 364 (2) (2010) 384–394.
- [9] S. Keshav, *An Engineering Approach to Computer Networking: ATM Networks, the Internet, and the Telephone Network*. Vol. 1, Addison-Wesley, 1997.
- [10] L. Kleinrock, *Queueing Systems*. Vol. 1, Wiley Interscience, 1975.
- [11] J. Kubasik, On some numerical methods for the computation of Erlang and Engset functions, in: *Proceedings of the Eleventh International Teletraffic Congress (ITC-11)*, Kyoto, Japan, 1985, pp. 958–964.
- [12] H. Le Vu, A. Zalesky, E.W.M. Wong, Z. Rosberg, S.M.H. Bilgrami, M. Zukerman, R.S. Tucker, Scalable performance evaluation of a hybrid optical switch, *J. Lightwave Technol.* 23 (10) (2005) 2961–2973.
- [13] A. Nail, Y. Gunalay, Stochastic analysis of finite population bufferless multiplexing in optical packet/burst switching systems, *IEICE Trans. Commun.* 90 (2) (2007) 342–345.
- [14] H. Overby, Performance modelling of optical packet switched networks with the Engset traffic model, *Opt. Express* 13 (5) (2005) 1685–1695.
- [15] R.S. Palais, A simple proof of the Banach contraction principle, *J. Fixed Point Theory Appl.* 2 (2) (2007) 221–223.
- [16] J.W. Pearson, *Computation of hypergeometric functions* (Master's thesis), University of Oxford, 2009.
- [17] M. Spivak, *Calculus*. Vol. 3, Publish or Perish, 1994.
- [18] J. Sztrik, Finite-source queueing systems and their applications, *Form. Methods Comput.* 1 (2001) 7–10.
- [19] H.C. Tijms, *A First Course in Stochastic Models*, John Wiley and Sons, 2003.
- [20] M. Zukerman, E.W.M. Wong, Z. Rosberg, G.M. Lee, H.L. Vu, On teletraffic applications to OBS, *IEEE Commun. Lett.* 8 (2) (2004) 116–118.